# Mathematical Analysis of the Navier-Stokes Equations with non standard boundary conditions

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**Abstract**

One of the major applications of the Domain Decomposition Time Marching Algorithm is the coupling of the Navier-Stokes systems with Boltzmann equations in order to compute transitional flows. Another important application, is the coupling of a global Navier-Stokes problem with a local one in order to use different modelizations and/or discretizations. Both of these applications involve a global Navier-Stokes systems with non standard boundary conditions. The purpose of this work is to prove, using the classical Leray-Schauder theory, that these boundary conditions are admissible and lead to a well posed problem.

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# 1 Introduction

In this paper we study the Navier-Stokes equations with the new boundary conditions introduced by the application of the Domain Decomposition Time Marching Algorithm ( [6], [9], and [10]). These boundary conditions are of slip type , they could appear either by the application of the Domain Decomposition Time Marching Algorithm to a Navier-Stokes problem ([9] and [10]), or to a Navier-Stokes/Boltzmann coupling ([7] and [9]). In the latter case, they are similar to the analytical slip boundary conditions introduced in [5], which are derived from kinetic theory in order to replace solving Boltzmann equations by solving Navier-Stokes equations in the transitional regime.

We study here only the stationary problem. The treatment of the time dependent problem will follow using the same ideas developed here (see [9]), and the classical proofs for the standard boundary conditions (see [1]-[3], and [4]). We begin by describing the strong formulation of the problem in the first paragraph, then we set the preliminary results necessary for our study. In the third paragraph we show the equivalence of the strong and the weak formulations under some regularity hypothesis, from which we deduce the admissible boundary conditions. We present, then, the study of the stationary problem. Paragraph 6, deals with the uniqueness results when the data are sufficiently small. Finally, we present, in the last paragraph, some conclusions.

# 2 Motivations

## 2.1 The general coupling strategy

For coupling external Navier-Stokes equations, with local Navier-Stokes equations (dense regimes) or local Boltzmann equations (transitional regimes), we introduce two domains, a global one $\Omega$, a local one $\Omega_V$ included in $\Omega$, and an interface $\Gamma_i$ (Fig. 1 in which $\Gamma_e$ denotes $\Gamma_\infty$). The global solution $W$ on $\Omega$ and the local solution $U_{loc}$ on $\Omega_V$, are matched by the following boundary

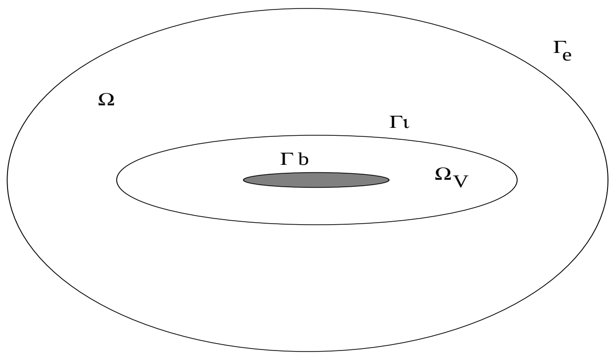


Figure 1: The global geometry

conditions, inspired of Schwarz overlapping techniques :

$$\begin{cases} W = \text{ given imposed value on } \Gamma_\infty, \\ n \cdot \sigma(W) \cdot \tau = n \cdot \sigma(U_{loc}) \cdot \tau \text{ on the body } \Gamma_o, \text{ (equality of friction forces)} \\ q(W) \cdot n + n \cdot \sigma(W) \cdot v = q(U_{loc}) \cdot n \text{ on } \Gamma_o, \text{ (equality of total heat fluxes)} \\ v \cdot n = 0 \text{ on } \Gamma_o, \\ U_{loc} = 0 \text{ on } \Gamma_o, \; U_{loc} = W \text{ on the interface } \Gamma_i. \end{cases}$$

Above, $n \cdot \sigma \cdot n$ and $n \cdot \sigma \cdot \tau$ respectively denote the normal and the tangential force exerted by the body on the flow, with $n$ the unit normal vector to the body oriented towards its interior.

The calculation of $U_{loc}$ and $W$ satisfying the above boundary conditions is then obtained by the Domain Decomposition Time Marching Algorithm, which was introduced by Le Tallec and Tidriri ([6], [9] and [10]) and which leads to the following algorithm :

**Initialization**

1. Guess an initial distribution of the conservative variable $W$ in the global domain $\Omega$ ;

2. Advance in time this distribution by using the global Navier-Stokes solver on $N_1$ time steps, with *Dirichlet* type boundary conditions on the body $\Gamma_o$ ;

3. Deduce from this result an initial distribution of the local variable $U_{loc}$ on the interface $\Gamma_i$ and in the local domain $\Omega_V$ ;

4. Advance in time this distribution by using the local solver on $N_2$ time steps with Dirichlet boundary conditions on $\Gamma_i$ and $\Gamma_o$.

**Iterations**

5. From $U_{loc}$, compute the friction forces $n \cdot \sigma(U_{loc}) \cdot \tau$ and heat flux $q(U_{loc}) \cdot n$ on the body $\Gamma_o$ ;

6. Advance the global solution in time ($N_1$ steps) by using the global Navier-Stokes solver with the above viscous forces as boundary conditions on $\Gamma_o$;

7. From $W$, compute the value of $U_{loc}$ on the interface $\Gamma_i$ ;

8. Using this new value as Dirichlet boundary conditions on $\Gamma_i$, advance the local solution in time ($N_2$ steps) and go back to step 5 until convergence is reached.

This algorithm completely uncouples the local and the global problems which can therefore be solved by independent solvers.

A parallel version is also quite possible although it is generally wiser to use parallel solvers within steps 6 and 8.

**Remark 2.1** *The local problem can be either Navier-Stokes or Boltzmann equations (see [6]-[10]).*

## 2.2 The global Navier-Stokes problem

The global domain $\Omega$ is discretized using node centered cells defined on an unstructured grid. Then, at each time step $n$ and for each cell $i$, we solve

$$\int_{C_i} \frac{W^{n+1} - W^n}{\Delta t} + \sum_{j \in V(i)} \int_{\partial C_i \cap \partial C_j} F_C(W^{n+1}) \cdot n_i$$

$$+ \int_{\partial C_i - \Gamma} F_D(W^{n+1}) \cdot n_i + \int_{\partial C_i \cap \Gamma_\infty} F(W^{n+1}) \cdot n_i = - \int_{\partial C_i \cap \Gamma_o} F_o \cdot n_i.$$

The fluxes $F_C$ and $F_D$ are computed at time step $n+1$ and linearized, with for example $F_C$ computed by an Osher approximate Riemann solver [9], and [10].

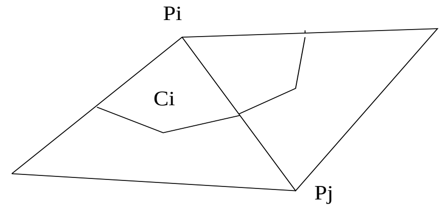


Figure 2: A boundary cell

On the body $\Gamma_o$, because of our special choice of boundary conditions, the flux is given by

$$\int_{\partial C_i \cap \Gamma_o} F_o \cdot n_i = \int_{\partial C_i \cap \Gamma_o} \begin{pmatrix} 0 \\ n_i \cdot \sigma(W^{n+1}) \cdot n_i \\ n_i \cdot \sigma(U_{loc}) \cdot \tau_i \\ -q(U_{loc}) \cdot n_i \end{pmatrix},$$

where the aspect of a boundary cell $C_i$ is described in Fig. 2.

In other words, friction forces and heat flux are given explicitly as predicted by the local solver and the mass flux is imposed to zero. Then, in order to have a well-posed problem, at least in the incompressible case (see next paragraphs), the normal stress (the multiplier of the zero mass flux constraint) cannot be imposed and must be obtained from the solution $W^{n+1}$.

**Remark 2.2** *Imposing friction forces to the global solution instead of no slip boundary conditions allows to have an accurate solution away from the boundary layer even with a coarse mesh (see [9]).*

# 3 Strong formulation

Let $\Omega$ be the domain occupied by the fluid, $\Gamma$ its boundary as described in figure (1); we assume that $\Omega$ satisfy :

$$\begin{cases} \Omega \text{ is a simply connected bounded domain in } \mathbb{R}^N, \\ \Gamma = \partial\Omega \text{ is of class } C^1, \\ \Gamma_b \cup \Gamma_\infty = \Gamma \quad \text{and} \quad \Gamma_b \cap \Gamma_\infty = \emptyset, \\ \Gamma_b \quad \text{and} \quad \Gamma_\infty \text{ are compacts of nonzero measure.} \end{cases} \tag{1}$$

In the steady incompressible case, the global problem in the coupling strategy described above consists in finding $u$ and $p$ satisfying the following equations :

$$-\nu\Delta u + u\nabla u + \nabla p = f \quad \text{in} \quad \Omega, \tag{2}$$

$$divu = 0 \quad \text{in} \quad \Omega, \tag{3}$$

$$u \cdot n = 0 \quad \text{on} \quad \Gamma_b, \tag{4}$$

$$u = h \quad \text{on} \quad \Gamma_\infty, \tag{5}$$

$$\nu(\nabla u \cdot n)\tau = g_\tau \quad \text{on} \quad \Gamma_b, \tag{6}$$

where $n$ is the unit normal to $\Gamma$, $\tau$ is any unit normal to $n$.

We look for weak solutions of problem (2)-(6), i.e. (2), (3) are satisfied in a distribution sense and equations (4)-(6) are satisfied in some Sobolev spaces.

**Remark 3.1** *From the remark (2.1) we observe that $g_\tau$ can be issued from a local kinetic model (Boltzmann equations) or a local continuous model (Navier-Stokes system).*

# 4 Preliminary results

Let $H^{m-\frac{1}{2}}(\Gamma)$ be a space of trace functions in $H^m(\Omega)$ with the following norm :

$$\|\psi\|_{m-\frac{1}{2},\Gamma} = \inf\{\|u\|_{m,\Omega}/u \in H^m(\Omega), u|_\Gamma = \psi\}, \tag{7}$$

and $V$ the closed subspace of $(H^1(\Omega))^N$ defined by

$$V = \{v \in (H^1(\Omega))^N | divv = 0,\ v \cdot n = 0 \quad \text{on} \quad \Gamma_b,\ v = 0 \quad \text{on} \quad \Gamma_\infty\}, \tag{8}$$

and define

$$\|v\| = \|\nabla v\|_{0,2,\Omega}. \tag{9}$$

We have the following classical lemma.

**Lemma 4.1** *$\|.\|$ is a norm of $V$ equivalent to the norm $\|.\|_{1,2,\Omega}$.*

**Lemma 4.2** *([11]) Under hypothesis (1), we have*

$$n: \quad \Gamma \quad \rightarrow \mathbb{R}^N \tag{10}$$

$$x \quad \rightarrow n(x)$$

*is $C^1$ and hence, the trace operator*

$$(H^1(\Omega))^N \rightarrow H^{\frac{1}{2}}(\Gamma) \tag{11}$$

$$u \rightarrow u \cdot n$$

*is continuous.*

**Lemma 4.3** *([14]). Let $\Omega$ satisfy (1). We have :*
*The injection*

$$W^{1,p}(\Omega) \subset L^{q_1}(\Omega)$$

*is compact for $q_1$ satisfying :*

$$1 \leq q_1 < \infty \quad \textit{if} \quad p \geq N \tag{12}$$

$$1 \leq q_1 < q \textit{ with } q \textit{ given by } \frac{1}{q} = \frac{1}{p} - \frac{1}{N} \quad \textit{if} \quad 1 \leq p < N. \tag{13}$$

In particular we have for $p = 2$ :

$$H^1(\Omega) \subset L^q(\Omega) \quad \text{for} \quad 2 \leq q \leq \frac{2N}{N-2},$$

the injection being continuous. For $N = 2$, this injection takes place for any finite $q$ such that $q \geq 2$.

**Lemma 4.4** *([12]) Let $\Omega$ be a bounded connected domain. Then*
*(i) The operator grad is an isomorphism from $L_0^2(\Omega)$ into $V^0$,*
*(ii) the operator div is an isomorphism from $(V^0)^\perp$ into $L_0^2(\Omega)$*
*where $L_0^2(\Omega)$ and $V^0$ are defined by*

$$V^0 = \{h \in H^{-1}(\Omega)^N, < h, v >= 0, \forall v \in V\}.$$

$$L_0^2(\Omega) = \{h \in L_0^2(\Omega), \int_\Omega h(x)dx = 0\}.$$

We now set

$$b(u,v,w) = \sum_{i,j=1}^{N} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx, u, v, w \in V \tag{14}$$

$$< f, v >= \int_\Omega f v dx, v \in V \tag{15}$$

$$< g_\tau, v >_{\Gamma_b}= \int_{\Gamma_b} g_\tau \tau \cdot v d\gamma, v \in V \tag{16}$$

$$((u,v)) = \int_\Omega \nabla u \nabla v dx, u, v \in V \tag{17}$$

$$(u,v) = \int_\Omega uv dx, u, v \in V. \tag{18}$$

If $f$ is in $(H^{-1}(\Omega))^N$ and $g_\tau$ in $H^{-\frac{1}{2}}(\Gamma_b)$ all the above operators (14)-(18) are well posed. Moreover, the operator defined by (16) and the trilinear form defined by (14) satisfy the following properties :

**Lemma 4.5** *The mapping from $V$ into $\mathbb{R}$ defined by (16) is continuous.*

**Proof:** The proof follows directly from lemma 4.2.

**Lemma 4.6** *(i) The trilinear form $b$ is continuous on $V^2 \times (V \cap (L^N(\Omega))^N$*

*(ii) Let $u, v, w \in V$. Then we have the following relations :*

$$b(u,v,w) = -b(u,w,v), \tag{19}$$

$$b(u,v,v) = 0. \tag{20}$$

*(iii) Let $\{u_n\}_{n\geq 0}$ be a sequence in $V$ such that $u_n \rightharpoonup u$ weakly in $V$. Then :*

$$\lim_{n\to\infty} b(u_n, u_n, v) = b(u,u,v), \forall v \in V. \tag{21}$$

**Proof**

We present only the proof of (i) and (ii). For the proof of (iii), see [14] or [13].

(i) Let $u \in V$ then $u_i \in H^1(\Omega)$ ; from lemma 4.3, we deduce :

$$u_i \in L^q(\Omega), \frac{1}{q} = \frac{1}{2} - \frac{1}{N},$$

which implies

$$\begin{aligned} |\int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i dx| &\leq \|u_j\|_{0,q,\Omega} \|\frac{\partial v_i}{\partial x_k}\|_{0,q,\Omega} \|w_i\|_{0,N,\Omega} \\ &\leq C\|u_j\|_V \|v_i\|_V \|w_i\|_V. \end{aligned} \tag{22}$$

Therefore we conclude to our result.

(ii) By definition

$$b(u,v,w) = \int_\Omega u_i v_{j,i} w_j.$$

After integration by parts, we obtain

$$\begin{aligned} b(u,v,w) &= -\int_\Omega v_j(u_i w_j)_{,i} + \int_{\partial\Omega} u \cdot n v \cdot w \\ &= -\int_\Omega v_j u_i w_{j,i} - \int_\Omega v_j w_j u_{i,i} + \int_{\partial\Omega} u \cdot n v \cdot w. \end{aligned}$$

Since $u \in V$, we have $u \cdot n = 0$ on $\partial\Omega$ and $u_{i,i} = div u = 0$ on $\Omega$. Therefore, we get

$$b(u,v,w) = -\int_\Omega v_j u_i w_{j,i} = -b(u,w,v).$$

**Remark 4.1** *For $N \leq 4$, it follows from lemma 4.3 that $V \cap (L^N(\Omega))^N = V$*

# 5 Existence Result

We now go back to our initial formulation:

Find $u$ and $p$ such that :

$$-\nu\Delta u + u \cdot \nabla u + \nabla p = f \quad \text{in} \quad \Omega, \tag{23}$$

$$div u = 0 \quad \text{in} \quad \Omega, \tag{24}$$

$$u \cdot n = \phi \quad \text{on} \quad \Gamma_b, \tag{25}$$

$$u = h \quad \text{on} \quad \Gamma_\infty, \tag{26}$$

$$\nu(\nabla u \cdot n)\tau = g_\tau \quad \text{on} \quad \Gamma_b. \tag{27}$$

## 5.1 Hypothesis on $\phi$ and $h$

The data $\phi$ on $\Gamma_b$ and $h$ on $\Gamma_\infty$ must verify some regularity and compatibility properties. More precisely we assume that :

$$h \in (H^{\frac{1}{2}}(\Gamma_\infty))^n, \tag{28}$$

$$\phi \in H^{\frac{1}{2}}(\Gamma_b), \tag{29}$$

$$\int_{\Gamma_\infty} h \cdot n = 0, \tag{30}$$

$$\int_{\Gamma_b} \phi = 0. \tag{31}$$

**Lemma 5.1** *The relations (28)-(31) imply the existence of $\tilde{u} \in (H^1(\Omega))^N$ such that :*

$$\nabla.\tilde{u} = 0 \quad in \quad \Omega, \tag{32}$$

$$\tilde{u} \cdot n = \phi \quad on \quad \Gamma_b, \tag{33}$$

$$\tilde{u} = h \quad on \quad \Gamma_\infty. \tag{34}$$

**Proof**
Let $\psi$ be a function defined on $\Gamma$ such that :

$$\psi = \begin{cases} \phi \cdot n & \text{on} \quad \Gamma_b, \\ h & \text{on} \quad \Gamma_\infty. \end{cases} \tag{35}$$

From the assumed hypothesis, on $\Gamma, \phi$ and $h$ we get :

$$\psi \in (H^{\frac{1}{2}}(\Gamma))^N.$$

Therefore, from the trace theorem, we can take $u \in H^1(\Omega)^N$ such that : $u|_\Gamma = \psi$. By using the relations (30) and (31) and the Green Formula, we obtain

$$\int_\Omega \nabla \cdot u = 0.$$

Using the regularity of $u$, we have $\nabla \cdot u \in L_0^2(\Omega)$.

Lemma 4.4 shows that there exists $v \in H_0^1(\Omega)^N$ such that $\nabla \cdot v = \nabla \cdot u$. Finally, by taking now $\tilde{u} = u - v$, we can see easily that $\tilde{u}$ verify the relations (32)-(34).

Now, we are going to pick out some function $\xi$ in $H^1(\Omega)$ such that :

$$\tilde{u} = rot\xi. \tag{36}$$

To achieve this, we will use the following proposition :

**Proposition 5.1** *([13]) Under the regularity hypothesis (1) and for $N = 3$ we have*

$$rotH^1(\Omega) = \{u \in L^2(\Omega)^N, divu = 0, \int_{\Gamma_b} u \cdot n = \int_{\Gamma_\infty} u \cdot n = 0\} \tag{37}$$

**Remark 5.1** *This result is also valid for $N = 2$ ([13]).*

Under the hypothesis (30)-(31) and the relations (32)-(34) we have :

$$\int_{\Gamma_b} \tilde{u} \cdot n = \int_{\Gamma_b} \phi = 0, \tag{38}$$

$$\int_{\Gamma_\infty} \tilde{u} \cdot n = \int_{\Gamma_b} h \cdot n = 0. \tag{39}$$

Consequently, from Proposition 5.1, there exists $\xi \in H^1(\Omega)$, such that we have (36). From now on we assume that $N \leq 3$.

## 5.2 Weak formulation

Let $u \in V$. After multiplication of (23) by $v$, integration by parts and by taking into account the boundary conditions (25)-(27) and the relation (24) we obtain the following variational formulation :

$$\begin{cases} \text{Find } u_o \in V \text{ such that} \\ \nu((u_o, v)) + b(\tilde{u}, u_o, v) + b(u_o, \tilde{u}, v) + b(u_o, u_o, v) \\ = -\nu((\tilde{u}, v)) - b(\tilde{u}, \tilde{u}, v) + \int_{\Gamma_b} g_\tau \tau \cdot v + \int_\Omega f v, \\ \forall v \in V \cap L^N(\Omega)^N \end{cases} \tag{40}$$

## 5.3 Equivalence between the two formulations

We will show here that the variational formulation (40) is equivalent to the strong formulation. This is the goal of the following proposition :

**Proposition 5.2** *If $(u,p)$ is a smooth solution of (2)-(6), then it satisfies (40). Conversely, if $u_o$ is solution of (40), then there exists a unique function (up to a constant) $p \in L^2(\Omega)$ such that $u_o + \tilde{u}$ and $p$ satisfy :*

**(i)** *(2) and (3) in a distribution sense,*

**(ii)** *(4) in $H^{\frac{1}{2}}(\Gamma_b)$,*

**(iii)** *(5) in $H^{\frac{1}{2}}(\Gamma_\infty)$,*

**(iv)** *(6) in $H^{-\frac{1}{2}}(\Gamma_b)$.*

**Proof**

The direct theorem was showed in our introduction of the weak formulation. We then show only the converse.

Let

$$\mathcal{V} = \{\psi \in (H_0^1(\Omega))^N / div\psi = 0\}.$$

Let $u_o$ be a solution of problem (40) then after multiplication by a test function in $\mathcal{D}(\Omega)$, and integration by parts and density, we get

$$< -\nu\Delta u + (u \cdot \nabla)u - f, \psi >= 0, \forall \psi \in \mathcal{V} \tag{41}$$

where $<,>$ denotes now the duality between the spaces $\mathcal{D}'(\Omega)^N$ and $\mathcal{D}(\Omega)^N$. We consider the operator $-grad(= -\nabla) \in \mathcal{L}(L_0^2(\Omega), H^{-1}(\Omega)^N)$, and $\mathcal{R}(-grad)$ its image in $H^{-1}(\Omega)^N$. We have the following lemma ( see [12])

**Lemma 5.2** *$\mathcal{R}(-grad)$ is identical to $\mathcal{V}^o$ with $\mathcal{V}^o$ defined by :*

$$\mathcal{V}^o = \{h \in H^{-1}(\Omega)^N / < h, \psi >= 0, \forall \psi \in \mathcal{V}\}.$$

The application of this lemma implies the existence of $p \in L^2_0(\Omega)$ such that

$$-\nu\Delta u + u \cdot \nabla u - f + \nabla p = 0 \quad \text{in} \quad (H^{-1}(\Omega))^N.$$

As $u \in V$, $u \cdot \nabla u$ and $f \in L^2$, then

$$\begin{aligned} -\nu\Delta u - \nabla p &= div(-\nu\nabla u + pId) \\ &= (f - u \cdot \nabla u) \in (L^2(\Omega))^N. \end{aligned}$$

Applying the Green formula implies that there exists

$$q = (-\nu\nabla u + pId) \cdot n \in H^{-\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} &\int_\Omega -div(-\nu\nabla u + pId) \cdot v \\ + \quad &\int_\Omega \nu\nabla u\nabla v - \int_\Omega pdivv = \int_{\partial\Omega} q \cdot v, \; \forall v \in (H^1(\Omega))^N. \end{aligned}$$

By replacing $div(-\nu\nabla u + pId)$ by $(f - u \cdot \nabla u)$ we get

$$\nu((u,v)) + b(u,u,v) - \int_\Omega pdivv - \int_{\partial\Omega} q \cdot v - \int_\Omega fv = 0 \qquad (42)$$
$$\forall v \in (H^1(\Omega))^N.$$

By subtracting (43) from (40), it remains

$$\int_{\partial\Omega} q \cdot v - \int_{\Gamma_b} g_\tau(v \cdot \tau) = 0, \; \forall v \in V.$$

Now let $w$ arbitrary element of $H^{\frac{1}{2}}_{00}(\Gamma_b)$, where $H^{\frac{1}{2}}_{00}(\Gamma_b)$ is defined by :

$$H^{\frac{1}{2}}_{00}(\Gamma_b) = \{v \in L^2(\Gamma_b); \exists w \in H^1(\Omega), w = 0 \text{ on } \Gamma_\infty \text{ and} w = v \text{ on } \Gamma_b\}$$

(see [11] for more details about this space).

Let $v \in V$ such that

$$v \cdot n = 0 \quad \text{on} \quad \Gamma_b,$$

$$v \cdot \tau = w \quad \text{on} \quad \Gamma_b,$$

$$v = 0 \quad \text{on} \quad \Gamma_\infty.$$

Applying the precedent equality, it remains

$$\int_{\Gamma_b} q \cdot \tau w = \int_{\Gamma_b} g_\tau w, \forall w \in H_{00}^{\frac{1}{2}}(\Gamma_b),$$

therefore

$$\begin{aligned} q \cdot \tau &= \tau \cdot (-\nu \nabla u + pId) \cdot n \\ &= g_\tau \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_b). \end{aligned}$$

**Remark 5.2** *We cannot impose the normal stress in (3)-(7) because of the presence of a pressure term in q.*

## 5.4 Existence result

Now, we are able to set out the main result of this section.

**Theorem 5.1** *Let $f \in H^{-1}(\Omega)$ ; then there exists $u_o \in V$ solution of (40).*

**Proof**
We proceed as in the usual proof of existence for the incompressible Navier-Stokes equations. Let us define

$$\begin{aligned} [\psi_m(u), u] &= \nu \|u\|^2 + \nu((\tilde{u}, u)) + b(\tilde{u}, \tilde{u}, u) - \int_{\Gamma_b} g_\tau \tau u \\ & - \int_\Omega fu + b(u, \tilde{u}, u). \end{aligned}$$

By using the continuity of $b$ we obtain :

$$b(\tilde{u}, \tilde{u}, u) \leq c_1 \|\tilde{u}\|_{0,q,\Omega} \|\tilde{u}\|_{1,2,\Omega} \|u\|_{0,N} \tag{43}$$

with $q$ satisfying (12)-(13). In addition we have :

$$b(\tilde{u}, \tilde{u}, u) \leq c_2 \|\tilde{u}\|^2_{1,2,\Omega} \|u\|. \tag{44}$$

Using lemma 4.1 and lemma 4.2 we arrive to :

$$\begin{aligned} [\psi_m(u), u] \quad \geq \quad & \nu \|u\|^2 + b(u, \tilde{u}, u) \\ & - \|u\| (\nu \|\tilde{u}\| + c_2 \|\tilde{u}\|^2_{1,2,\Omega} + c_3 \|g_\tau\|_{-\frac{1}{2},\Gamma_b} + \|f\|_{-1,2,\Omega}). \end{aligned}$$

So, to be able to use the Brouwer fixed point theorem, it is sufficient to show that we have for some $\beta > 0$, the following relation :

$$\nu \|u\|^2 + b(u, \tilde{u}, u) \geq \beta \|u\|^2, \forall u \in V. \tag{45}$$

This is the goal of the following lemma :

**Lemma 5.3** $\forall \ \gamma > 0$ *we can choose* $\tilde{u}$ *verifying (32)-(33) such that*

$$|b(u, \tilde{u}, u)| \leq \gamma \|u\|^2. \tag{46}$$

From the above lemma, we conclude to the existence of a given $k$ such that

$$[\psi_m(u), (u)] > 0,$$

for any $u \in V$ with $\|u\| = k$. Then the Brouwer fixed point theorem can be applied to the function $\psi_m(u)$ inside any ball of radius $k$ belonging to any given finite dimensional approximation $V_m$ of $V$. From this, we deduce the existence of a Galerkin approximation $u_m \in V_m$ of the solution $u$ of (40). By standard compactness arguments, it follows that $u_m$ weakly converges to a solution $u_o$ of (40) in $V$. (see [5] for more details).

## 6 Uniqueness results

Under the assumption that $N \leq 3$, we have the following result :

**Theorem 6.1** *Assume that $\tilde{u}$ in $(L^N(\Omega))^N$ is sufficiently small so that*

$$|b(v, \tilde{u}, v)| \leq \frac{\nu}{2}\|v\|^2, \forall v \in V, \tag{47}$$

*and $\nu$ is sufficiently large so that*

$$\nu^2 > 4C(\|\tilde{f}\|_{V'} + \|g_\tau\|_{-\frac{1}{2},\Gamma_b}) \tag{48}$$

*where $C$ is the constant in (22) and*

$$\tilde{f} = \nu\Delta\tilde{u} - \Sigma_j \tilde{u}_j \frac{\partial}{\partial x_j}\tilde{u} + f$$

*and*

$$\tilde{g}_\tau = \nu\frac{\partial}{\partial n}\tilde{u} + g_\tau$$

**Proof**

We procced as in the proof for the standard homogeneous boundary conditions (see for instance [13]).

Suppose that $u_o$ is solution of the problem (40). Taking $v = u_o$ in (40) and using (20), we obtain :

$$\nu\|u_o\|_V^2 = -b(u_o, \tilde{u}, u_o) - \nu((\tilde{u}, u_o)) - b(\tilde{u}, \tilde{u}, u_o) + \int_{\Gamma_b} g_\tau \tau \cdot u_o + \int_\Omega f u_o, \tag{49}$$

Using the Green formula, we obtain

$$\nu\|u_o\|_V^2 = -b(u_o, \tilde{u}, u_o) + < \nu\Delta\tilde{u} - \Sigma_j \tilde{u}_j \frac{\partial}{\partial x_j}\tilde{u} + f, u_o > \tag{50}$$

$$+ < \nu\frac{\partial}{\partial n}\tilde{u} + g_\tau, u_o >_{\Gamma_b} \tag{51}$$

$$= -b(u_o, \tilde{u}, u_o) + < \tilde{f}, u_o > + < \tilde{g}_\tau, u_o >_{\Gamma_b} \tag{52}$$

Since $f \in (H^{-1}(\Omega))^N$ and $\tilde{u} \in (H^1(\Omega))^N$, $\Delta\tilde{u} \in (H^{-1}(\Omega))^N$ and then $\tilde{f} \in (H^{-1}(\Omega))^N$. Similarly, since $g_\tau \in (H^{-\frac{1}{2}}(\Gamma_b))^N$ and $\tilde{u} \in (H^1(\Omega))^N$ we have $\dfrac{\partial\tilde{u}}{\partial n} \in (H^{-\frac{1}{2}}(\Gamma_b))^N$ and then $\tilde{g}_\tau \in (H^{-\frac{1}{2}}(\Gamma_b))^N$, and the equality above is well defined. The rest of the proof is an adaptation of the proof for the standard homogeneous boundary conditions (see for instance [13]).

# 7 Conclusion

In this paper we have shown that the slip boundary conditions resulting from the application of the Domain Decomposition Time Marching Algorithm to either Navier-Stokes/Navier-Stokes coupling ([9] and [10]) or Navier-Stokes/Boltzmann coupling ([9], [7], and [8]) are admissible and lead to a well posed global problem at least in the incompressible case.

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